From multiple scattering to van der Waals interactions: exact results for eccentric cylinders

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In this paper, dedicated to the career of Iver Brevik, we review the derivation of the retarded van der Waals or Casimir-Polder interaction between polarizable molecules from the general multiple scattering formulation of Casimir interactions between bodies. We then apply this van der Waals potential to examine the interaction between tenuous cylindrical bodies, including eccentric cylinders, and revisit the vanishing self-energy of a tenuous dielectric cylinder. In each case, closed-form expressions are obtained.

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I. INTRODUCTION

Since the earliest calculations of fluctuation forces between bodies [1], that is, Casimir or quantum vacuum forces, multiple scattering methods have been employed. Rather belatedly, it has been realized that such methods could be used to obtain accurate numerical results in many cases [2, 3, 4, 5]. These results allow us to transcend the limitations of the proximity force theorem (PFT) [6, 7], and so make better comparison with experiment, which typically involve curved surfaces. (For a review of the experimental situation, see Ref. [8].)

These improvements in technique were inspired in part by the development of the numerical Monte-Carlo worldline method of Gies and Klingmüller [9, 10, 11, 12] but the difficulty with this latter method lies in the statistical limitations of Monte Carlo methods and in the complexity of incorporating electromagnetic boundary conditions. Optical path approximations have been studied extensively for many years, with considerable success [13, 14, 15, 16]. However, there always remain uncertainties because of unknown errors in excluding diffractive effects. Direct numerical methods [17, 18], based on finite-difference engineering techniques, may have promise, but the requisite precision of 3-dimensional calculations may prove challenging [19].

The multiple scattering formalism, which is in principle exact, dates back at least into the 1950s [20, 21]. Particularly noteworthy is the seminal work of Balian and Duplantier [22]. (For more complete references see Ref. [23].) This technique, which has been brought to a high state of perfection by Emig et al. [5], has concentrated on numerical results for the Casimir forces between conducting and dielectric bodies such as spheres and cylinders. Recently, we have noticed that the multiple-scattering method can yield exact, closed-form results for bodies that are weakly coupled to the quantum field [23, 24]. This allows an exact assessment of the range of applicability of the PFT. The calculations there, however, as those in recent extensions of our methodology [25], have been restricted to scalar fields with δ-function potentials, so-called semitransparent bodies. (These are closely related to plasma shell models [3, 26, 27, 28].) The technique was recently extended to dielectric bodies [29], characterized by a permittivity ε. Strong coupling would mean a perfect metal, $\varepsilon \to \infty$, while weak coupling means that ε is close to unity.

In this paper we will give details of the formalism, and show how in weak coupling (dilute dielectrics) we recover the sum of Casimir-Polder or retarded van der Waals forces between atoms. Exact results have been found in the past in such summations, for example for the self-energy of a dilute dielectric sphere [30] or a dilute dielectric cylinder [31]. Thus it is not surprising that exact results for the interaction of different dilute bodies can be obtained. It is only surprising that such results were not found much earlier. (We note that the additive approximation has been widely used in the past, for example, see Ref. [32], but here the method is exact. Also, there are many exact computations for non-retarded London forces betwen bodies, e.g., Ref. [33], but these results can only apply to very tiny objects on the nanometer scale.) In our previous letter [29] we considered the force and torque on a slab of finite extent above an infinite plane, and the force between spheres and parallel cylinders outside each other. Here we will examine further

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cylindrical geometries, such as concentric cylinders, and eccentric circular cylinders with parallel axes, in cases where the dielectric materials do not overlap. We will prove that the results can be obtained by analytic continuation of the energies found earlier for non-contained bodies. Finally, we will re-examine the self-energy of a dielectic cylinder [31].

II. GREEN'S DYADIC FORMALISM

For electromagnetism, we can start from the formalism of Schwinger [34], which is based on the electric Green's dyadic $\mathbf{\Gamma}$. This object can be identified as the one-loop vacuum expectation value of the correlation function of electric fields,

$$\mathbf{\Gamma}(\mathbf{r},t;\mathbf{r}',t') = i\langle \mathrm{T}\{\mathbf{E}(\mathbf{r},t)\mathbf{E}(\mathbf{r}',t')\}\rangle. \tag{2.1}$$

Alternatively, we regard the Green's dyadic as the propagator between a polarization source $\mathbf{P}$ and a phenomenological field $\mathbf{E}$ (where $x^\mu = (\mathbf{r},t)$):

$$\mathbf{E}(x) = \int (dx')\mathbf{\Gamma}(x,x)\cdot\mathbf{P}(x'). \tag{2.2}$$

We will only be contemplating static geometries, so it is convenient to consider a specific frequency ω, as introduced through a Fourier transform,

$$\mathbf{\Gamma}(x,x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')}\mathbf{\Gamma}(\mathbf{r},\mathbf{r}';\omega), \tag{2.3}$$

in terms of which the Maxwell equations in a region where the permittivity $\varepsilon(\omega)$ and the permeability $\mu(\omega)$ are constant in space read

$$\boldsymbol{\nabla}\times\mathbf{\Gamma} = i\omega\mathbf{\Phi}, \quad \boldsymbol{\nabla}\cdot\mathbf{\Phi} = 0, \tag{2.4a}$$

$$\frac{1}{\mu}\boldsymbol{\nabla}\times\mathbf{\Phi} = -i\omega\varepsilon\mathbf{\Gamma'}, \quad \boldsymbol{\nabla}\cdot\mathbf{\Gamma'} = 0, \tag{2.4b}$$

where we have introduced $\mathbf{\Gamma'} = \mathbf{\Gamma} + \mathbf{1}/\varepsilon$, where the unit dyadic includes a spatial δ function. The two Green's dyadics given here satisfy the following inhomogenous Helmholtz equations,

$$(\nabla^2 + \omega^2\varepsilon\mu)\mathbf{\Gamma'} = -\frac{1}{\varepsilon}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{1}), \tag{2.5a}$$

$$(\nabla^2 + \omega^2\varepsilon\mu)\mathbf{\Phi} = i\omega\mu\boldsymbol{\nabla}\times\mathbf{1}. \tag{2.5b}$$

In the following, it will prove more convenient to use, instead of Eq. (2.5a),

$$\left(\frac{1}{\omega^2\mu}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times -\varepsilon\right)\mathbf{\Gamma} = \mathbf{1}. \tag{2.6}$$

In the presence of a polarization source, the action is, in symbolic form,

$$W = \frac{1}{2}\int \mathbf{P}\cdot\mathbf{\Gamma}\cdot\mathbf{P}, \tag{2.7}$$

so if we consider the interaction between bodies characterized by particular values of ε and μ, the change in the action due to moving those bodies is

$$\delta W = \frac{1}{2}\int \mathbf{P}\cdot\delta\mathbf{\Gamma}\cdot\mathbf{P} = -\frac{1}{2}\int \mathbf{E}\cdot\delta\mathbf{\Gamma}^{-1}\cdot\mathbf{E}, \tag{2.8}$$

where the symbolic inverse dyadic, in the sense of Eq. (2.6), is

$$\mathbf{\Gamma}^{-1} = \frac{1}{\omega^2\mu}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times -\varepsilon, \tag{2.9}$$

that is,

$$\delta\mathbf{\Gamma}\cdot\mathbf{\Gamma}^{-1} = -\mathbf{\Gamma}\cdot\delta\mathbf{\Gamma}^{-1}. \tag{2.10}$$

By comparing with the iterated source term in the vacuum-to-vacuum persistence amplitude $\exp iW$, we see that an infinitesimal variation of the bodies results in an effective source product,

$$\mathbf{P}(\mathbf{r})\mathbf{P}(\mathbf{r}')\Big|_{\text{eff}} = i\delta\mathbf{\Gamma}^{-1}, \tag{2.11}$$

from which we deduce from Eq. (2.7) that

$$\delta W = \frac{i}{2}\text{Tr}\,\mathbf{\Gamma}\cdot\delta\mathbf{\Gamma}^{-1} = -\frac{i}{2}\text{Tr}\,\delta\mathbf{\Gamma}\cdot\mathbf{\Gamma}^{-1} = -\frac{i}{2}\delta\text{Tr}\,\ln\mathbf{\Gamma}, \tag{2.12}$$

where the trace includes integration over space-time coordinates. We conclude, by ignoring an integration constant,

$$W = -\frac{i}{2}\text{Tr}\,\ln\mathbf{\Gamma}. \tag{2.13}$$

This is in precise analogy to the expression for scalar fields. Another derivation of this result is given in the Appendix. Incidentally, note that the first equality in Eq. (2.12) implies for dielectric bodies ($\mu = 1$)

$$\delta W = -\frac{i}{2}\int\frac{d\omega}{2\pi}\int(d\mathbf{r})\,\delta\varepsilon(\mathbf{r},\omega)\Gamma_{kk}(\mathbf{r},\mathbf{r}';\omega), \tag{2.14}$$

which is the starting point for the derivation of the Lifshitz formula [35] in Ref. [34].

III. REDERIVATION OF CASIMIR-POLDER FORMULA

Henceforth, let us consider pure dielectrics, that is, set $\mu = 1$. The free Green's dyadic, in the absence of dielectric bodies, satisfies the equation

$$\left[\frac{1}{\omega^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times -1\right]\mathbf{\Gamma}_0 = \mathbf{1}, \tag{3.1}$$

so the equation satisfied by the full Green's dyadic is

$$(\mathbf{\Gamma}_0^{-1} - V)\mathbf{\Gamma} = \mathbf{1}, \tag{3.2}$$

where $V = \varepsilon - 1$ within the body. From this we deduce immediately that

$$\mathbf{\Gamma} = (1 - \mathbf{\Gamma}_0 V)^{-1}\mathbf{\Gamma}_0. \tag{3.3}$$

From the trace-log formula (2.13) we see that the energy for a static situation ($W = -\int dt\, E$) relative to the free-space background is

$$E = \frac{i}{2}\text{Tr}\,\ln\mathbf{\Gamma}_0^{-1}\cdot\mathbf{\Gamma} = -\frac{i}{2}\text{Tr}\,\ln(1 - \mathbf{\Gamma}_0 V). \tag{3.4}$$

The trace here is only over spatial coordinates. We will now consider the interaction between two bodies, with non-overlapping potentials, $V = V_1 + V_2$, where $V_a = \varepsilon_a - 1$ is confined to the interior of body a, $a = 1, 2$. Although it is straightforward to proceed to write the interaction between the bodies in terms of scattering operators, for our limited purposes here, we will simply treat the potentials as weak, and retain only the first, bilinear term in the interaction:

$$E_{12} = \frac{i}{2}\text{Tr}\,\mathbf{\Gamma}_0 V_1 \mathbf{\Gamma}_0 V_2. \tag{3.5}$$

Here, as may be verified by direct calculation [36],

$$\mathbf{\Gamma}_0(\mathbf{r},\mathbf{r}') = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{1}G_0(\mathbf{r}-\mathbf{r}') - \mathbf{1} = (\boldsymbol{\nabla}\boldsymbol{\nabla} - \mathbf{1}\zeta^2)G_0(\mathbf{r}-\mathbf{r}'), \tag{3.6}$$

where the scalar Helmholtz Green's function which satisfies causal or Feynman boundary conditions is

$$G_0(\mathbf{r}-\mathbf{r}') = \frac{e^{-|\zeta|R}}{4\pi R}, \quad R = |\mathbf{r}-\mathbf{r}'|, \tag{3.7}$$

the Fourier transform of the Euclidean Green's function, which obeys the differential equation

$$(-\nabla^2 + \zeta^2) G_0(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \tag{3.8}$$

and $\zeta = -i\omega$.

Thus the interaction between the two potentials is given by

$$E_{12} = -\frac{1}{2} \int \frac{d\zeta}{2\pi} \int (d\mathbf{r})(d\mathbf{r}') \left[(\nabla_i \nabla_j - \zeta^2 \delta_{ij}) \frac{e^{-|\zeta||\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \right]^2 V_1(\mathbf{r}) V_2(\mathbf{r}'). \tag{3.9}$$

The derivatives occurring here may be easily worked out:

$$(\mathbf{\Gamma}_0)_{ij} = (\nabla_i \nabla_j - \zeta^2 \delta_{ij}) \frac{e^{-|\zeta|R|}}{4\pi R} = \left[-\delta_{ij}(1 + |\zeta|R + \zeta^2 R^2) + \frac{R_i R_j}{R^2}(3 + 3|\zeta|R| + \zeta^2 R^2) \right] \frac{e^{-|\zeta|R}}{4\pi R^3}, \tag{3.10}$$

and then contracting two such factors together gives

$$(\nabla_i \nabla_j - \zeta^2 \delta_{ij}) \frac{e^{-|\zeta|R}}{4\pi R} (\nabla_i \nabla_j - \zeta^2 \delta_{ij}) \frac{e^{-|\zeta|R}}{4\pi R} = (6 + 12t + 10t^2 + 4t^3 + 2t^4) \frac{e^{-2t}}{(4\pi R^3)^2}, \tag{3.11}$$

where $t = |\zeta|R$. Inserting this into Eq. (3.9), we obtain for the integral over ζ

$$-\frac{1}{64\pi^3 R^7} \int_0^\infty du\, e^{-u} \left(6 + 6u + \frac{5}{2}u^2 + \frac{1}{2}u^3 + \frac{1}{8}u^4 \right) = -\frac{23}{64\pi^3 R^7}, \tag{3.12}$$

or

$$E_{12} = -\frac{23}{(4\pi)^3} \int (d\mathbf{r})(d\mathbf{r}') \frac{V_1(\mathbf{r}) V_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^7}, \tag{3.13}$$

which is the famous Casimir-Polder potential [37]. This formula is valid for bodies, which are presumed to be composed of material filling nonoverlapping volumes v_1 and v_2, respectively, characterized by dielectric constants ε_1 and ε_2, both nearly unity. We emphasize that this formula is exact in the limit $\varepsilon_{1,2} \to 1$, as discussed in Ref. [32].

IV. ENERGY OF CYLINDER PARALLEL TO A PLANE

In Ref. [29] we derived the energy of two uniform dilute cylinders, of radius a and b respectively, the parallel axes of which are separated by a distance R, $R > a + b$. In terms of the constant

$$N = \frac{23}{640\pi^2}(\varepsilon_1 - 1)(\varepsilon_2 - 1). \tag{4.1}$$

the energy of interaction per unit length is

$$\mathfrak{E}_{\text{cyl}-\text{cyl}} = -\frac{32\pi N}{3} \frac{a^2 b^2}{R^6} \frac{1 - \frac{1}{2}\left(\frac{a^2+b^2}{R^2}\right) - \frac{1}{2}\left(\frac{a^2-b^2}{R^2}\right)^2}{\left[\left(1 - \left(\frac{a+b}{R}\right)^2\right)\left(1 - \left(\frac{a-b}{R}\right)^2\right)\right]^{5/2}}. \tag{4.2}$$

If we take R and b to infinity, such that $Z = R - b$ is held fixed, we describe a cylinder of radius a parallel to a dielectric plane, where Z is the distance between the axis of the cylinder and the plane. That limit gives the simple result

$$\mathfrak{E}_{\text{cyl}-\text{pl}} = -\frac{N\pi a^2}{Z^4} \frac{1}{(1 - a^2/Z^2)^{5/2}}. \tag{4.3}$$

This is to be compared to the corresponding result for a sphere of radius a a distance Z above a plane:

$$E_{\text{sph}-\text{pl}} = -N \frac{v}{Z^4} \frac{1}{(1 - a^2/Z^2)^2}, \tag{4.4}$$

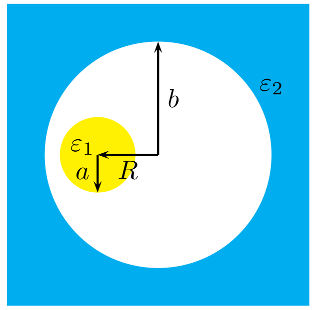

FIG. 1: Dielectric ε_2 hollowed out by a cylindrical cavity which contains an offset parallel dielectric cylinder ε_1.

where v is the volume of the sphere.

As an illustration of how the calculation is done, let us rederive this result directly from Eq. (3.13). We see immediately that the energy between an infinite halfspace (of permittivity ε_1) and a parallel slab (of permittivity ε_2) of area A and thickness dz separated by a distance z is

$$\frac{dE}{A} = -N\frac{dz}{z^4}, \tag{4.5}$$

so the energy per length between the cylinder and the plane is

$$\mathfrak{E}_{\rm cyl-pl} = -2Na^2\int_{-1}^{1} d\cos\theta\frac{\sin\theta}{(Z+a\cos\theta)^4} = -\frac{N\pi a^2 Z}{(Z^2-a^2)^{5/2}}, \tag{4.6}$$

which is the result (4.3).

V. ECCENTRIC CYLINDERS

As a second illustration, consider two coaxial cylinders, of radii a and b, $a<b$. The inner cylinder is filled with material of permittivity ε_1, while the outer cylinder is the inner boundary of a region with permittivity ε_2 extending out to infinity. An easy calculation from the van der Waals interaction (3.13)

$$\begin{aligned}\mathfrak{E}_{\rm co-cyl} &= -\frac{64N}{3}\int_0^a d\rho\,\rho\int_b^\infty d\rho'\rho'\int_0^{2\pi}\frac{d\theta}{(\rho^2+\rho'^2-2\rho\rho'\cos\theta)^3}\\ &= -\frac{32\pi N}{3}\int_0^{a^2}dx\int_{b^2}^\infty dy\frac{x^2+y^2+4xy}{(y-x)^5}\\ &= -\frac{16N\pi a^2b^2}{3(b^2-a^2)^3}.\end{aligned} \tag{5.1}$$

This reduces to the dilute Lifshitz formula for the interaction between parallel plates if we take the limit $b\to\infty$, $a\to\infty$, with $b-a=d$ held fixed:

$$\mathfrak{E}_{\rm co-cyl}\to -\frac{2N\pi b}{3d^3},\quad \text{or}\quad \frac{E}{A} = -\frac{N}{3d^3}. \tag{5.2}$$

Note that the result (5.1) may be obtained by analytically continuing the energy between two externally separated cylinders, given by Eq. (4.2). We simply take R to zero there, and choose the sign of the square root so that the energy is negative. That suggests that the same thing can be done to obtain the energy of interaction between two parallel cylinders, one inside the other, but whose axes are displaced by an offset R, with $R+a<b$, as shown in Fig. 1:

$$\mathfrak{E}_{\rm ecc-cyl} = -\frac{16\pi N}{3}\frac{a^2}{b^4}\frac{(1-a^2/b^2)^2+(1+a^2/b^2)R^2/b^2-2R^4/b^4}{\left[(1-a^2/b^2)^2+R^4/b^4-2(1+a^2/b^2)R^2/b^2\right]^{5/2}}. \tag{5.3}$$

We can verify this is true by carrying out the integral from the Casimir-Polder formula (3.13)

$$\mathfrak{E}_{\mathrm{ecc-cyl}} = -\frac{32N}{3\pi}\int_0^a \rho\, d\rho \int_b^\infty \rho'\, d\rho' \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \left[\rho^2 + \rho'^2 - 2\rho\rho'\cos(\theta-\theta') + R^2 - 2R(\rho\cos\theta - \rho'\cos\theta')\right]^{-3}. \tag{5.4}$$

The angular integrals can be done, but the remaining integrals are rather complicated. Therefore, let us simply expand immediately in the quantities $x = \rho/\rho'$ and $y = R/\rho'$, which are both less than one. Then we can carry out the four integrals term by term. In this way we find

$$\mathfrak{E}_{\mathrm{ecc-cyl}} = -\frac{16\pi N a^2}{3b^4}\sum_{n,m=0}^\infty \left(\frac{a^2}{b^2}\right)^n \left(\frac{R^2}{b^2}\right)^m \frac{(m+1)^2}{2}\binom{n+m+1}{m+1}\binom{n+m+2}{m+1}, \tag{5.5}$$

which is exactly the series expansion of Eq. (5.3) for small a/b and R/b.

By differentiating this energy with respect to the offset R, we obtain the force of interaction between the inner cylinder and the outer one, $\mathfrak{F} = -\partial\mathfrak{E}/\partial R$. Evidently, that force is zero for coaxial cylinders, since that is a point of unstable equilibrium. For small R, $\mathfrak{F}$ grows linearly with R with a positive coefficient. The inner cylinder is attracted to the closest point of the opposite cylinder. Similar considerations for conducting cylinders were given in Refs. [38, 39], with the idea that the cylindrical geometry might prove to be a useful proving ground for Casimir experiments.

VI. SELF-ENERGY OF DILUTE CYLINDER

This is a rederivation of the result found by dimensional continuation in Ref. [31]. The summation of the Casimir-Polder forces between the molecules in a single cylinder of radius a is given by (in $N\ \varepsilon_1 = \varepsilon_2$)

$$\begin{aligned}\mathfrak{E}_{\mathrm{cyl}} &= -\frac{32N}{3}\int_0^a d\rho\,\rho \int_0^a d\rho'\,\rho' \int_0^{2\pi} \frac{d\theta}{(\rho^2+\rho'^2-2\rho\rho'\cos\theta)^3} \\ &= -\frac{16N}{3}\int_0^{a^2}\frac{dx}{x^2}\int_0^1 du\left(\frac{1}{u^3}-\frac{6}{u^4}+\frac{6}{u^5}\right).\end{aligned} \tag{6.1}$$

This is, of course, terribly divergent. We can regulate it by analytic continuation: replace the highest power of $(\rho^2-\rho'^2)^{-1} = (xu)^{-1}$, 5, by β, and regard β as less than 1. Then,

$$\begin{aligned}\mathfrak{E}_{\mathrm{cyl}} &= -\frac{16N}{3}\int_0^{a^2} dx\, x^{3-\beta}\int_0^1 du\left(u^{2-\beta}-6u^{1-\beta}+6u^{-\beta}\right) \\ &= -\frac{16N}{3}(a^2)^{4-\beta}\frac{(5-\beta)}{(1-\beta)(2-\beta)(3-\beta)}.\end{aligned} \tag{6.2}$$

Now if we analytically continue to $\beta = 5$ we get an vanishing self-energy to order $(\varepsilon-1)^2$. This result was first discovered by Romeo (private communication), verified in Ref. [31], and confirmed later by full Casimir calculations [40, 41].

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APPENDIX A: DERIVATION OF GREEN'S DYADIC FORMALISM FROM CANONICAL THEORY

Here we begin by sketching the development of the Green's dyadic equation from canonical quantum electrodynamics. For simplicity of the discussion, we will consider a medium without dispersion, so that ε and μ are constant. First

we must state the canonical equal-time commutation relations. We will require (only transverse fields are relevant)

$$[E_i(\mathbf{r},t), E_j(\mathbf{r}',t)] = 0. \tag{A1}$$

In a medium, it is the electric displacement field which is canonically conjugate to the vector potential, so we have the equal-time commutation relation (Coulomb gauge)

$$[\mathbf{A}(\mathbf{r},t), \partial_0 \mathbf{A}(\mathbf{r}',t)] = \frac{i}{\varepsilon}\left(\mathbf{1} - \frac{\boldsymbol{\nabla}\boldsymbol{\nabla}}{\nabla^2}\right)\delta(\mathbf{r}-\mathbf{r}'). \tag{A2}$$

Now in view of Eq. (2.1), and the Maxwell equations

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \tag{A3a}$$

$$\boldsymbol{\nabla}\times\frac{1}{\mu}\mathbf{B} = \frac{\partial}{\partial t}\varepsilon\mathbf{E}, \tag{A3b}$$

we deduce from Eq. (2.1)

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\Gamma'} + \varepsilon\mu\frac{\partial^2}{\partial t^2}\boldsymbol{\Gamma'} = i\varepsilon\mu\delta(t-t')\langle[\dot{\mathbf{E}}(\mathbf{r},t), \mathbf{E}(\mathbf{r}',t)]\rangle. \tag{A4}$$

But according to Maxwell's equations and Eq. (A2)

$$[\dot{\mathbf{E}}(\mathbf{r},t), \mathbf{E}(\mathbf{r}',t)] = \frac{1}{\varepsilon\mu}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times[\mathbf{A}(\mathbf{r},t), -\partial_0\mathbf{A}(\mathbf{r}',t)] = -\frac{i}{\varepsilon^2\mu}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{1}\delta(\mathbf{r}-\mathbf{r}'). \tag{A5}$$

If we now insert this into Eq. (A4) we obtain for the Fourier transform of the Green's dyadic (2.3)

$$(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times-\varepsilon\mu\omega^2)\boldsymbol{\Gamma'} = \frac{1}{\varepsilon}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{1}, \tag{A6}$$

which is indeed the equation satisfied by the solenoidal Green's dyadic, Eq. (2.5a).

It is equally easy to derive the trace-log formula. We have the variational statement, for infinitesimal changes in the permittivity and the permeability [42],

$$\delta E = -\frac{1}{2}\int(d\mathbf{r})\langle\delta\varepsilon E^2 + \delta\mu H^2\rangle. \tag{A7}$$

Given Eqs. (2.1) and (A3a) we can write this in terms of the coincident-point limit of Green's dyadic,

$$\delta E = \frac{i}{2}\int(d\mathbf{r})\left[\delta\varepsilon - \delta\left(\frac{1}{\mu}\right)\frac{1}{\omega^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\right]\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}')\bigg|_{\mathbf{r}'\to\mathbf{r}} = -\frac{i}{2}\mathrm{Tr}\,\delta\boldsymbol{\Gamma}^{-1}\cdot\boldsymbol{\Gamma}, \tag{A8}$$

according to Eq. (2.9), which involves an integration by parts, and coincides with the first equality in Eq. (2.12).

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